MY NOTATION, BUT FOLLOWING SHREVE pp. 195–196

LET $\Pi(t) = V(t) - S(t)\cdot\Delta(t)$ BE THE <u>HEDGE</u> PORTFOLIO.

USE ITÔ PRODUCT RULE (COROLLARY 4.6.3)

THEN

$$d\Pi(t) = dV(t) - d\,(S(t)\cdot\Delta(t))$$

$$= \left(\frac{\partial V}{\partial t}dt + \cancel{\frac{\partial V}{\partial S}dS} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\cdot dt\right)$$

$$- \left(S(t)\cdot d\Delta + \cancel{dS\,\Delta(t)} + dS\cdot d\Delta\right)$$

$\Delta \equiv \frac{\partial V}{\partial S}$

$$\therefore\ d\Pi(t) = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt - \left(S(t)\cdot d\Delta + dS\cdot d\Delta\right) \quad (*)$$

LET $X(t)$ BE THE <u>REPLICATING</u> STRATEGY FOR $V(t)$, WHERE $X(t)$ HOLDS $\Delta(t) = \frac{\partial V}{\partial S}$ SHARES OF STOCK, WITH BORROWING OR LENDING IN THE RISKLESS ASSET, AS NEEDED, TO SELF-FINANCE THE POSITION.

THE <u>INVESTMENT</u> IN THE RISKLESS ASSET (SHREVE CALLS IT A MONEY MARKET A/C) IS THUS

$$X(t) - \Delta(t)\cdot S(t) = V(t) - \Delta(t)\cdot S(t) = \Pi(t).$$

LET $M(t) = e^{rt}$ BE THE PRICE OF ONE SHARE OF THE RISKLESS ASSET. THEN THE <u>NUMBER</u> OF SHARES IN THE RISKLESS ASSET USED IN THE REPLICATING STRATEGY IS

$$\Gamma(t) = \frac{\Pi(t)}{M(t)} = e^{-rt}\,\Pi(t)$$

THE CONTINUOUS-TIME SELF-FINANCING CONDITION (4.10.15) IS

$$S(t)\cdot d\Delta + dS\cdot d\Delta + M(t)\cdot d\Gamma + dM\cdot d\Gamma = 0. \quad (**)$$

WE WANT TO PLUG IN FOR M, dM, & $d\Gamma$.

$$M(t) = e^{rt} \Rightarrow dM = re^{rt}\,dt = rM(t)\,dt$$

$$\Gamma(t) = e^{-rt}\,\Pi(t)$$

$$\Rightarrow d\Gamma = -re^{-rt}\,\Pi(t)\,dt + e^{-rt}\,d\Pi - re^{-rt}\,dt\,d\Pi$$

$$= -r\frac{\Pi(t)}{M(t)}\,dt + \frac{d\Pi}{M(t)} - r\frac{dt\cdot d\Pi}{M(t)}$$

SO THE CTSF CONDITION (**) YIELDS

$$S(t)\,d\Delta + dS\cdot d\Delta + M(t)\left[-r\frac{\Pi(t)}{M(t)}\,dt + \frac{d\Pi}{M(t)} - r\frac{dt\cdot d\Pi}{M(t)}\right]$$

$$+ rM(t)\,\cancel{dt}\left(-r\frac{\Pi(t)}{M(t)}\,\cancel{dt} + \frac{d\Pi}{M(t)} - r\frac{\cancel{dt}\cdot d\Pi}{M(t)}\right) = 0$$

$$\therefore\ S(t)\,d\Delta + dS\cdot d\Delta - \boxed{r\,\Pi(t)\,dt} + d\Pi - r\,dt\cdot\cancel{d\Pi} + r\cancel{d}t\,d\Pi = 0$$

SO, USING (*), WE GET

$$\cancel{S(t)\cdot d\Delta + dS\cdot d\Delta} - \boxed{r\,\Pi(t)\,dt} + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt$$

$$-\left(\cancel{S(t)\cdot d\Delta + dS\cdot d\Delta}\right) = 0$$

AND THUS $r\,\Pi(t)\,dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt$ (4.10.18).

NOW PLUG IN $\Pi(t) = V(t) - S(t)\cdot\Delta(t) = V(t) - S(t)\frac{\partial V}{\partial S}$

AND CANCEL OUT dt FROM EACH SIDE TO GET

$$r\left(V - \frac{\partial V}{\partial S}S\right) = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2},$$

AND REARRANGE TO GET

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} - rV = 0 \quad \left(\begin{array}{c}\text{B.SCHOLES}\\ \text{PDE}\end{array}\right)$$

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